UNIVERSITÉ DU LUXEMBOURG
ANALYSE 1
2015-2016

EXERCISE SHEET 4

4.1. Compute the following limits:

4.1.1. $\lim_{x\to-\infty} \frac{x^3+2\sqrt{x}}{3x^2-\sqrt{x}}$

4.1.2. $\lim_{x\to 0+} \frac{x^3+2\sqrt{x}}{3x^2-\sqrt{x}}$

4.1.3. $\lim_{x\to 0+} \frac{x^2+2^x}{3x^2+3^x}$

4.1.4. $\lim_{x\to+\infty} \frac{x^2+2^x}{3x^2+3^x}$

4.1.5. $\lim_{x\to 0+} \frac{x^2-2\log(x)}{3x^2+\log(x)}$

4.1.6. $\lim_{x\to+\infty} \frac{x^2-2\log(x)}{3x^2+\log(x)}$

4.1.7. $\lim_{x\to\pi-} \frac{x^2+\cos(x)}{x-\pi}$

4.1.8. $\lim_{x\to+\infty} \frac{x^2+\cos(x)}{x-\pi}$

4.1.9. $\lim_{x\to 0+} \sin\left(\frac{1}{x}\right)$

4.1.10. $\lim_{x\to+\infty} \sin(x)\sin\left(\frac{1}{x}\right)$

4.1.11. $\lim_{x\to 2+} \frac{\log(x^2-3)}{x^4-x-14}$

4.1.12. $\lim_{x\to\infty} \frac{\log(x^2-3)}{x^4-x-14}$

4.2. Compute the limit of the following functions for $x \to 0$

4.2.1. $f(x) = \frac{\exp(3x)-1}{x}$

4.2.2. $f(x) = \frac{\sin^2(3x^3)}{x^6}$

4.2.3. $f(x) = \frac{\exp(x)-\cos(2x)}{x}$

4.2.4. $f(x) = \frac{\cos(x+x^2)-1}{x\sin(x)+x\log(1+2x)}$

4.2.5. $f(x) = (\cos(x))^{\frac{1}{\sin^2(x)}}$

4.2.6. $f(x) = \frac{\sqrt{2+x}-\sqrt{2+3x}}{\sqrt{3+x}-\sqrt{3+3x}}$

4.2.7. $f(x) = \frac{\sqrt{\cos(x)}-1}{\sqrt[3]{x^2+1}-\exp(x)}$

4.3. Let $a, b, c \in \mathbb{R}^+$. Compute the following limits:

4.3.1. $\lim_{x\to+\infty}\left(a+\frac{b}{x}\right)^{ax+c}$

4.3.2. $\lim_{x\to+\infty}\left(\frac{a^x+b^x}{2}\right)^{\frac{1}{x}}$

4.3.3. $\lim_{x\to-\infty}\left(\frac{a^x+b^x}{2}\right)^{\frac{1}{x}}$

4.4. We say that a function $f : (a, b) \to \mathbb{R}$ is Lipschitz if there exists a constant $C > 0$ such that for every $x, y \in (a, b)$ the following relation holds

$$|f(x) - f(y)| \leq C|x - y| \ .$$

4.4.1. Show that every Lipschitz function is continuous.

4.4.2. Show that if f, g are Lipschitz, the sum $(f + g)(x) := f(x) + g(x)$ is Lipschitz.

4.4.3. Show that if f, g are Lipschitz and the composition $f \circ g$ is well-defined, then $f \circ g$ is Lipschitz.

4.4.4. Show that the affine functions $f(x) = ax + b$ for $a, b \in \mathbb{R}$ are Lipschitz.

4.4.5. Give an example of a Lipschitz function $f : (0, 1) \to (0, 1)$, which is not affine.
Hint: Look for some polynomials